

Fermionic Meixner probability distributions, Lie algebras and quadratic Hamiltonians

L. Accardi (1), I.Ya. Aref'eva, I.V.Volovich (2)

(1) Centro Vito Volterra
Università di Roma "Tor Vergata"
Roma I-00133 Italy

(2) Steklov Mathematical Institute,
Gubkin St.8, GSP-1, 117966, Moscow, Russia,
volovich@mi.ras.ru, arefeva@mi.ras.ru

Contents

1	Introduction	2
1.1	Triviality of the 1-mode quadratic Fermi algebra	4
1.2	The quadratic Fermi algebra	4
1.3	Vacuum expectations of quadratic Fermi Hamiltonians	6
2	1-mode Bose quadratic Hamiltonians	10
3	The Meixner distributions	17
3.1	The 5-th Meixner class	17
3.2	The 4-th Meixner class: Gamma distributions	18
3.3	The 3-d Meixner class: Negative Binomial (Pascal) distribu- tions	18
4	Meixner distributions and quadratic Bose Hamiltonians	19
5	n-dimensional case	25
5.1	Vector valued Meixner random variables	27
5.2	The case of commuting A and C	28
5.3	The 1-dimensional case	29

Abstract We introduce the quadratic Fermi algebra, which is a Lie algebra, and show that the vacuum distributions of the associated Hamiltonians define the fermionic Meixner probability distributions. In order to emphasize the difference with the Bose case, we apply a modification of the method used in the above calculation to obtain a simple and straightforward classification of the 1-dimensional Meixner laws in terms of homogeneous quadratic expressions in the Bose creation and annihilation operators. There is a huge literature of the Meixner laws but this, purely quantum probabilistic, derivation seems to be new. Finally we briefly discuss the possible multi-dimensional extensions of the above results.

1 Introduction

The program of **quadratic quantization** was introduced in the paper [7] where the realizability of this program was proved by explicit construction of the Fock representation of the quadratic Bose algebra (see Definition (3) below). In the paper [6] the current algebra of the quadratic Bose algebra over \mathbb{R}^d was identified with the corresponding current algebra of $sl(2, \mathbb{R})$. This allowed to include the problem of constructing representations of the quadratic Bose algebra into the general theory of factorizable representations of Lie algebras (see [15] and the bibliography therein) and its extension to quantum Levy processes on \ast -bi-algebras (see [8], [16]). Combining these techniques with the fact that the irreducible unitary representations of $sl(2, \mathbb{R})$ are classified, in [6] a class of unitary representation of the quadratic Bose algebra was built and the vacuum distributions of the generalized field operators were identified with the 3 non standard (i.e. non Gauss or Poisson) Meixner probability distributions.

The problem of the existence of a Fermi analogue of the quadratic Bose algebra was investigated in [1] where:

- (i) a natural candidate for the role of **quadratic Fermi algebra** was constructed (see Definition (1) below);
- (ii) the Lie algebra isomorphism between the quadratic Fermi algebra and $sl(2, \mathbb{R})$ was proved (this in particular implies the Lie algebra isomorphism between the quadratic Fermi and Bose algebras);
- (iii) it was shown that the quadratic Fermi and Bose algebras cannot be \ast -isomorphic for the natural involutions induced by their explicit construc-

tions in terms of Fermions or Bosons respectively.

The non $*$ -isomorphism result, mentioned in item (iii) above, naturally rises the problem of identifying the vacuum distributions of the field operators in the Fock representation of the quadratic Fermi algebra. This problem is solved in section 1.3 of the present paper. From this one can deduce a weaker form of the above mentioned non $*$ -isomorphism result, namely that there cannot exist a $*$ -isomorphism between the two algebras mapping the Fermi quadratic annihilator into a multiple of its Bose analogue. In fact, if such a $*$ -isomorphism existed, then the restriction of the Fock states of the two representations to the Cartan sub-algebras of the two algebras should give rise to the same class of probability measures. However, comparing the results of section 1.3 below with those of [6] (see also section 2 below) one immediately verifies that this is not the case.

In section 2 below it is shown that the method developed for the calculation of the Fermi vacuum distributions can be applied to the Bose case leading to a very simple and elegant classification of all possible vacuum distributions of homogeneous quadratic Boson Hamiltonians in terms of Meixner laws. On both of these topics, **separately**, there exists a vast literature [10], [9] for vacuum distributions of quadratic expressions of usual Boson fields (where however the connection with Meixner classes was missing) and [12], [13] for the probabilistic aspects of Meixner laws and the connection between the two was established in [6] using the theory of orthogonal polynomials. The method used here is different and we have reasons to believe that it can be extended to the multi-dimensional case (in this case the orthogonal polynomial method can only be applied to products of Meixner measures). The problems related with the multi-dimensional case are briefly outlined in the final section 5 .

We emphasize that the present paper is only the beginning of our investigation the quadratic Fermi algebra. In this direction there are many deep problems which have not yet an answer. Among them we mention:

- (I) the existence of the Fock representation the current algebra over \mathbb{R}^d of the quadratic Fermi algebra (for this the methods of [6] should be sufficient);
- (II) in case of existence, the vacuum distribution of the corresponding fields;
- (III) the validity, also in the Fermi case, of the no-go theorems proved in [17], [6], [3], [4].

These problems are now under investigation.

Finally let us mention that, after completing the present paper, we received the very interesting preprint [18] including a different approach to the notion of multi-dimensional Meixner random variables, based on the quantum decomposition of a classical multi-dimensional random variable with all moments and on its characterization in terms of commutators (see [2]). This naturally poses the problem to clarify the mutual relationships between these two approaches.

1.1 Triviality of the 1-mode quadratic Fermi algebra

Recall that the 1-mode CAR algebra, denoted $CAR(1)$, is the $*$ -algebra with identity 1, generators $\{a, a^+, 1\}$ called the **Fermi creation and annihilation operators** and relations

$$(a^+)^* = a$$

$$\{a, a\} = \{a^+, a^+\} = 0 \quad ; \quad \{a, a^+\} := aa^+ + a^+a = 1 \quad (1)$$

As a consequence of (1), one has

$$a^2 = a^{+2} = 0$$

which implies that both a^+a and aa^+ are orthogonal projections. In fact they are clearly self-adjoint and: Because of (1) the pair $\{aa^+, a^+a\}$ is a partition of the identity. Thus, for a 1-mode representation, the quadratic algebra

$$\{a^2, a^{+2}, aa^+\} = \{0, aa^+\}$$

is generated by the single projection, hence it is abelian.

1.2 The quadratic Fermi algebra

A possible candidate for the role of (non-trivial) quadratic Fermi algebra is constructed as follows. Consider the 2-mode CAR algebra, denoted $CAR(2)$, i.e. the associative $*$ -algebra with identity 1_F , generators: a_i, a_j^+ ($i, j = 1, 2$) and relations

$$a_j = (a_j^+)^* \quad ; \quad \{a_i, a_j\} = \{a_i^+, a_j^+\} = 0 \quad ; \quad i, j = 1, 2 \quad (2)$$

$$\{a_i, a_j^+\} := a_i a_j^+ + a_j^+ a_i =: [a_i, a_j^+]_+ = \delta_{ij} \cdot 1_F$$

Lemma 1 In the above notations, defining

$$F^- := a_1 a_2$$

$$F^+ := a_2^+ a_1^+ = (F^-)^*$$

$$N_F = N_F^* := a_2^+ a_2 + a_1^+ a_1$$

the following **quadratic Fermi commutation relations** hold:

$$[F^-, F^+] = -N_F + 1_F =: M_F \quad (3)$$

$$[N_F, F^+] = [M_F, F^+] = 2F^+ \quad (4)$$

$$[N_F, F^-] = [M_F, F^-] = -2F^- \quad (5)$$

$$[X, 1_F] = 0 \quad ; \quad X = F^\pm, \quad (6)$$

Remark Notice that $N_F \neq 1_F$, in fact for example

$$a_1^+ N_F = a_1^+ (a_2^+ a_2 + a_1^+ a_1) = a_1^+ a_2^+ a_2 = a_2^+ a_2 a_1^+ \neq N_F a_1^+$$

Therefore the right hand side of (3) is not equal to zero. Denoting

$$S_1 := F^+ \quad ; \quad S_2 := F^- \quad ; \quad S_3 := N_F \quad ; \quad S_0 := 1_F$$

(3), (4), (5) become equivalent to

$$[S_1, S_2] = -S_3 + S_0 \quad ; \quad [S_3, S_1] = -2S_1 \quad ; \quad [S_3, S_2] = 2S_2 \quad (7)$$

$$[S_0, X] = 0$$

$$(S_1)^* = S_2 \quad ; \quad (S_3)^* = S_3 \quad ; \quad (S_0)^* = S_0 \quad (8)$$

Definition 1 *-Lie algebra with generators (S_1, S_2, S_3, S_0) and relations (7), (8) is called the quadratic Fermi algebra.

We will consider a representation of the quadratic Fermi algebra in a Hilbert space with the vacuum vector ψ_0 such that $a_j \psi_0 = 0, j = 1, 2$.

1.3 Vacuum expectations of quadratic Fermi Hamiltonians

The following problem naturally arises: *Which are the vacuum distributions of the field operators of the quadratic Fermi algebra?*

To discuss this problem we define, for $\alpha, \beta \in \mathbb{R}$, the homogenous quadratic Hamiltonian:

$$H = \alpha(a_1^+ a_1 + a_2^+ a_2) + \beta(a_1^+ a_2^+ + a_2 a_1) \quad (9)$$

and the associated time evolution, uniquely determined, on the polynomial $*$ -algebra generated by the a_i, a_i^+ ($i = 1, 2$), as the solutions of the equations

$$\partial a_j(t) := i[H, a_j(t)] \quad ; \quad a_j(0) = a_j, \quad j = 1, 2 \quad (10)$$

Lemma 2 Suppose α, β are arbitrary real numbers, such that

$$\omega = \sqrt{\alpha^2 + \beta^2} > 0 \quad (11)$$

Then the Heisenberg evolutions of a_i, a_i^+ , $i = 1, 2$ are given by:

$$a_1(t) = \left(\cos \omega t - \frac{i\alpha}{\omega} \sin \omega t \right) a_1 - \frac{i\beta}{\omega} \sin \omega t \cdot a_2^+ \quad (12)$$

$$a_2(t) = \left(\cos \omega t - \frac{i\alpha}{\omega} \sin \omega t \right) a_2 + \frac{i\beta}{\omega} \sin \omega t \cdot a_1^+ \quad (13)$$

$$a_1^+(t) = \left(\cos \omega t + \frac{i\alpha}{\omega} \sin \omega t \right) a_1^+ + \frac{i\beta}{\omega} \sin \omega t \cdot a_2 \quad (14)$$

$$a_2^+(t) = \left(\cos \omega t + \frac{i\alpha}{\omega} \sin \omega t \right) a_2^+ - \frac{i\beta}{\omega} \sin \omega t \cdot a_1 \quad (15)$$

Proof. Note that

$$\begin{aligned} i[H, a_1] &= i[\alpha a_1^+ a_1 + \beta a_1^+ a_2^+, a_1] = \\ &= i\alpha a_1^+ a_1 a_1 + i\beta a_1^+ a_2^+ a_1 - i\alpha a_1 a_1^+ a_1 - i\beta a_1 a_1^+ a_2^+ = \\ &= -i\beta a_1^+ a_1 a_2^+ - i\alpha a_1 + i\beta a_1^+ a_1 a_2^+ - i\beta a_2^+ = -i\alpha a_1 - i\beta a_2^+ \\ i[H, a_2] &= i[\alpha a_2^+ a_2 + \beta a_1^+ a_2^+, a_2] = \end{aligned} \quad (16)$$

$$\begin{aligned}
&= i\alpha a_2^+ a_2 a_2 + i\beta a_1^+ a_2^+ a_2 - i\alpha a_2 a_2^+ a_2 - i\beta a_2 a_1^+ a_2^+ = \\
&= i\beta a_1^+ a_2^+ a_2 - i\alpha a_2 - i\beta a_1^+ a_2^+ a_2^+ a_2 + i\beta a_1^+ = -i\alpha a_2 + i\beta a_1^+
\end{aligned} \tag{17}$$

Now note that

$$\begin{aligned}
\frac{da_1(t)}{dt} &= i[H, a_1(t)] = i[H, e^{itH} a_1 e^{-itH}] = e^{itH} i[H, a_1] e^{-itH} = \\
&= e^{itH} (-i\alpha a_1 - i\beta a_2^+) e^{-itH} = -i\alpha a_1(t) - i\beta a_2^+(t)
\end{aligned}$$

Similarly,

$$\frac{da_2(t)}{dt} = -i\alpha a_2(t) + i\beta a_1^+(t)$$

It is easy to deduce the equations for $\frac{da_1^+(t)}{dt}$ and $\frac{da_2^+(t)}{dt}$. The result is summarized in the following equation:

$$\frac{d}{dt} \begin{pmatrix} a_1(t) \\ a_2(t) \\ a_1^+(t) \\ a_2^+(t) \end{pmatrix} = i \begin{pmatrix} -\alpha & 0 & 0 & -\beta \\ 0 & -\alpha & \beta & 0 \\ 0 & \beta & \alpha & 0 \\ -\beta & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} a_1(t) \\ a_2(t) \\ a_1^+(t) \\ a_2^+(t) \end{pmatrix} =: iJ \begin{pmatrix} a_1(t) \\ a_2(t) \\ a_1^+(t) \\ a_2^+(t) \end{pmatrix} \tag{18}$$

with initial condition $a_j^\pm(0) = a_j^\pm$. Therefore, the solution of this differential equation is:

$$\begin{pmatrix} a_1(t) \\ a_2(t) \\ a_1^+(t) \\ a_2^+(t) \end{pmatrix} = \exp \left(it \begin{pmatrix} -\alpha & 0 & 0 & -\beta \\ 0 & -\alpha & \beta & 0 \\ 0 & \beta & \alpha & 0 \\ -\beta & 0 & 0 & \alpha \end{pmatrix} \right) \begin{pmatrix} a_1 \\ a_2 \\ a_1^+ \\ a_2^+ \end{pmatrix}$$

In the notation (18) J^2 is equal to

$$\begin{aligned}
&\begin{pmatrix} -\alpha & 0 & 0 & -\beta \\ 0 & -\alpha & \beta & 0 \\ 0 & \beta & \alpha & 0 \\ -\beta & 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} -\alpha & 0 & 0 & -\beta \\ 0 & -\alpha & \beta & 0 \\ 0 & \beta & \alpha & 0 \\ -\beta & 0 & 0 & \alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 + \beta^2 & 0 & 0 & 0 \\ 0 & \alpha^2 + \beta^2 & 0 & 0 \\ 0 & 0 & \alpha^2 + \beta^2 & 0 \\ 0 & 0 & 0 & \alpha^2 + \beta^2 \end{pmatrix} \\
&= (\alpha^2 + \beta^2) \cdot 1 = \omega^2 \cdot 1
\end{aligned}$$

Therefore

$$J^{2n+1} = \omega^{2n} J \quad ; \quad J^{2n} = \omega^{2n}$$

and from this one verifies that the matrix exponential is

$$\exp(itJ) = \begin{pmatrix} \cos \omega t - \frac{i\alpha}{\omega} \sin \omega t & 0 & 0 & -\frac{i\beta}{\omega} \sin \omega t \\ 0 & \cos \omega t - \frac{i\alpha}{\omega} \sin \omega t & \frac{i\beta}{\omega} \sin \omega t & 0 \\ 0 & \frac{i\beta}{\omega} \sin \omega t & \cos \omega t + \frac{i\alpha}{\omega} \sin \omega t & 0 \\ -\frac{i\beta}{\omega} \sin \omega t & 0 & 0 & \cos \omega t + \frac{i\alpha}{\omega} \sin \omega t \end{pmatrix}$$

Hence, we get Eqs. (12-15).

Lemma 3 Denote

$$f(t) := \langle \psi_0, e^{itH} \psi_0 \rangle \quad (19)$$

Then f satisfies the equation

$$\left(1 - \frac{\beta^2}{\omega^2} \sin^2 \omega t\right) \frac{df}{dt} = -\frac{\beta^2}{2\omega} \sin \omega t \left(\cos \omega t + \frac{i\alpha}{\omega} \sin \omega t\right) f \quad (20)$$

with ω given by (11) and

$$f(0) = 1 \quad (21)$$

Proof. Let $f(t)$ be given by (19). We have

$$\frac{df}{dt} = i\beta \langle \psi_0, e^{itH} a_1^+ a_2^+ \psi_0 \rangle = i\beta \langle \psi_0, a_2 a_1 e^{itH} \psi_0 \rangle$$

Let us use the identity:

$$e^{itH} a_1^+ a_2^+ = e^{itH} a_1^+ I a_2^+ I = e^{itH} a_1^+ e^{-itH} e^{itH} a_2^+ e^{-itH} e^{itH} = a_1^+(t) a_2^+(t) e^{itH} \quad (22)$$

Substituting $a_1^+(t) a_2^+(t)$ from Lemma 2, we get:

$$\begin{aligned} \frac{df}{dt} &= i\beta \langle \psi_0, a_1^+(t) a_2^+(t) e^{itH} \psi_0 \rangle = \\ &= i\beta \left\langle \psi_0, \frac{i\beta}{\omega} \sin \omega t \cdot a_2 \left(\left(\cos \omega t + \frac{i\alpha}{\omega} \sin \omega t \right) a_2^+ - \frac{i\beta}{\omega} \sin \omega t \cdot a_1 \right) e^{itH} \psi_0 \right\rangle \\ &= -\frac{\beta^2}{\omega} \sin \omega t \left(\cos \omega t + \frac{i\alpha}{\omega} \sin \omega t \right) \langle \psi_0, a_2 a_2^+ e^{itH} \psi_0 \rangle \\ &\quad + i\beta \frac{\beta^2}{\omega^2} \sin^2 \omega t \langle \psi_0, a_2 a_1 e^{itH} \psi_0 \rangle \end{aligned} \quad (23)$$

Since

$$\langle \psi_0, a_2 a_2^+ e^{itH} \psi_0 \rangle = \langle \psi_0, (1 - a_2^+ a_2) e^{itH} \psi_0 \rangle = \langle \psi_0, e^{itH} \psi_0 \rangle = f(t)$$

and according to (23),

$$\langle \psi_0, a_2 a_1 e^{itH} \psi_0 \rangle = \frac{1}{i\beta} \frac{df}{dt}$$

we get:

$$\frac{df}{dt} = -\frac{\beta^2}{\omega} \sin \omega t \left(\cos \omega t + \frac{i\alpha}{\omega} \sin \omega t \right) f + \frac{\beta^2}{\omega^2} \sin^2 \omega t \frac{df}{dt}$$

which is (20). The initial condition (50) is obvious.

Lemma 4

$$\langle \psi_0, e^{itH} \psi_0 \rangle = \left(\cos \omega t - \frac{i\alpha}{\omega} \sin \omega t \right) e^{i\alpha t} \quad (24)$$

Proof. By direct calculations one can check that the function (24) is the solution of the differential equation (20) with the initial condition (21).

Theorem 1 *The vacuum expectation $f(t) = \langle \psi_0, e^{itH} \psi_0 \rangle$ is the characteristic function of the Bernoulli random variable X with distribution*

$$p_X(x) = \frac{1}{2} \left(\left(1 - \frac{\alpha}{\omega}\right) \delta(x + \alpha + \omega) + \left(1 + \frac{\alpha}{\omega}\right) \delta(x + \alpha - \omega) \right) \quad (25)$$

Proof. This follows from Lemma 4. In fact:

$$\begin{aligned} F^{-1} \left[\left(\cos \omega t - \frac{i\alpha}{\omega} \sin \omega t \right) e^{i\alpha t} \right] &= F^{-1} \left[\left(\frac{1}{2} (e^{i\omega t} + e^{-i\omega t}) - \frac{\alpha}{2\omega} (e^{i\omega t} - e^{-i\omega t}) \right) e^{i\alpha t} \right] \\ &= \frac{1}{2} F^{-1} \left[\left(1 - \frac{\alpha}{\omega}\right) e^{i(\omega+\alpha)t} + \left(1 + \frac{\alpha}{\omega}\right) e^{i(\alpha-\omega)t} \right] = \\ &= \frac{1}{2} \left(\left(1 - \frac{\alpha}{\omega}\right) \delta(x + \alpha + \omega) + \left(1 + \frac{\alpha}{\omega}\right) \delta(x + \alpha - \omega) \right). \end{aligned}$$

Remark. Notice that the limit of (25) as $\omega \rightarrow 0$ in such a way that $\alpha/\omega \rightarrow c \in \mathbb{R}$ exists and (independently of c) is equal to

$$p_{0,X}(x) = \delta(x) \quad (26)$$

Definition 2 *For varying α and β the family (25) can be easily seen to coincide with the class of all Bernoulli distributions. The class corresponding to $\alpha \neq \beta$ will be called the 1-st Fermionic Meixner classes. The 2-d Fermionic Meixner class, corresponding to $\alpha = \beta$, consists of the single δ -measure (26).*

2 1–mode Bose quadratic Hamiltonians

In this section we recall some known facts about quadratic 1–mode Hamiltonians. Let $\{\mathcal{H}, a^+, a, \psi_0\}$ be the (unique up to unitary equivalence) Fock representation of the 1–mode canonical commutation relations (CCR). This means that \mathcal{H} is a Hilbert space, with scalar product denoted by $\langle \cdot, \cdot \rangle$, $\psi_0 \in \mathcal{H}$ is a unit vector, called the vacuum, and a^+, a are operators on \mathcal{H} , called respectively creation and annihilation operators satisfying the Fock relation

$$a\psi_0 = 0$$

and such that, for each $n \in \mathbb{N}$, ψ_0 is in the domain of $(a^+)^n$, the linear span of the vectors $(a^+)^n \psi_0$ is dense in \mathcal{H} (in this case we say that ψ_0 is cyclic for the polynomial algebra generated by a^+ and a) and on this domain the 1–mode CCR

$$[a, a^+] = 1 \tag{27}$$

are satisfied.

Definition 3 The 4–dimensional complex $*$ –Lie algebra with generators (T_1, T_2, T_3, T_0) and relations

$$[T_2, T_1] = T_3 \quad ; \quad [T_3, T_1] = 2T_1 \quad ; \quad [T_3, T_2] = -2T_2 \quad ; \quad [T_0, X] = 0 \tag{28}$$

$$(T_1)^* = T_2 \quad ; \quad (T_3)^* = T_3 \quad ; \quad (T_0)^* = T_0 \tag{29}$$

is called the quadratic Bose algebra

Remark. In the notation (27), the identifications:

$$T_1 := a^{+2} \quad ; \quad T_2 := a^2 \quad ; \quad T_3 := 4a^+a + 2$$

give a concrete realization of the quadratic Bose algebra on the space of the Fock representation of the Heisenberg algebra. Note the difference with Schwinger bosons. Schwinger uses a doublet of bosons (see for example [19], [20]) while we use a singlet.

The most general symmetric, real homogeneous, quadratic expression in the variables a^+, a has the form:

$$H := \frac{1}{2}\alpha (a^+)^2 + \frac{1}{2}\bar{\alpha}a^2 + \beta a^+a \tag{30}$$

where $\alpha \in \mathbb{C}$, $\beta \in \mathbb{R}$. H is not identically zero if and only if

$$|\alpha|^2 + \beta^2 > 0 \quad (31)$$

Introducing the matrix

$$h := \begin{pmatrix} \alpha & \beta \\ \beta & \bar{\alpha} \end{pmatrix}$$

H can be represented, up to an inessential additive constant, in the form:

$$H = \frac{1}{2} + \frac{1}{2}(a^+, a)h \begin{pmatrix} a^+ \\ a \end{pmatrix}$$

Without loss of generality it can be assumed that

$$\alpha \in \mathbb{R}_+ \quad (32)$$

in fact if $\alpha = |\alpha|e^{i\theta}$, where $\theta \in \mathbb{R}$, then the gauge transformation

$$a \rightarrow \tilde{a} := e^{-i\theta/2}a \quad ; \quad a^+ \rightarrow \tilde{a}^+ := e^{i\theta/2}a^+$$

leaves invariant the commutation relations (27) and the quadratic Hamiltonian H in terms of \tilde{a} and \tilde{a}^+ becomes:

$$H = \frac{1}{2}|\alpha| \left((\tilde{a}^+)^2 + \tilde{a}^2 \right) + \beta \tilde{a}^+ \tilde{a}$$

H generates the 1-parameter unitary group $U_t = e^{itH}$ and the Heisenberg evolution of any operator A under this group (we can simply say under H) is defined by

$$A(t) := e^{itH} A e^{-itH} \quad (33)$$

Lemma 5 Let $\alpha, \beta \in \mathbb{R}$ satisfy (31) and let ω denote the positive square root of

$$\omega^2 := \beta^2 - \alpha^2 = -\det h \quad (34)$$

Then, if $\omega \neq 0$ the Heisenberg evolutions of a, a^+ under H are given by:

$$a(t) = \left(\cos \omega t - \frac{i\beta}{\omega} \sin \omega t \right) a - \frac{i\alpha}{\omega} \sin \omega t \cdot a^+ \quad (35)$$

$$a^+(t) = \left(\cos \omega t + \frac{i\beta}{\omega} \sin \omega t \right) a^+ + \frac{i\alpha}{\omega} \sin \omega t \cdot a \quad (36)$$

If $\omega = 0$ they are given by the limit of the expressions (35) and (36) as $\omega \rightarrow 0$, i.e.:

$$a(t) = (1 - i\beta t) a - i\alpha t a^+ \quad (37)$$

$$a^+(t) = (1 + i\beta t) a + i\alpha t a^+ \quad (38)$$

Remark. Notice that the evolution (35), (36), hence also (37), (38), are invariant under change of sign of ω . In particular they do not depend on the choice of the real square root of ω .

Proof. Note that

$$\begin{aligned} \frac{da(t)}{dt} &= i[H, a(t)] = i[H, e^{itH} a e^{-itH}] = i e^{itH} [H, a] e^{-itH} = \\ &= i e^{itH} (-\beta a - \alpha a^+) e^{-itH} = -i\beta a(t) - i\alpha a^+(t) \end{aligned} \quad (39)$$

Similarly,

$$\frac{da^+(t)}{dt} = i\beta a^+(t) + i\alpha a(t) \quad (40)$$

With the notations $a_t \equiv a(t)$, $\dot{a}_t \equiv da(t)/dt$, from (39) we get

$$i\alpha a_t^+ = -i\beta a_t - \dot{a}_t \quad (41)$$

or equivalently, for

$$\begin{aligned} \alpha &\neq 0 \\ a_t^+ &= -\frac{\beta}{\alpha} a_t - \frac{1}{i\alpha} \dot{a}_t \end{aligned} \quad (42)$$

Taking the time derivative of this equation and using (40), (41) we get

$$-\frac{\beta}{\alpha} \dot{a} - \frac{1}{i\alpha} \ddot{a} = \dot{a}^+ = i\beta a^+ + i\alpha a = -\frac{i\beta^2}{\alpha} a - \frac{\beta}{\alpha} \dot{a} + i\alpha a \quad (43)$$

$$-\frac{1}{i\alpha} \ddot{a} = \frac{\beta^2}{i\alpha} a + i\alpha a \quad (44)$$

Therefore

$$\ddot{a} + (\beta^2 - \alpha^2) a = 0 \quad (45)$$

In the notation (34) we write equation (45) in the form

$$\ddot{a} + \omega^2 a = 0 \quad (46)$$

Since $U_0 = 1$, $a(0) = a$ and $a^+(0) = a^+$. From (39) we get the initial conditions:

$$a(0) = a, \quad \dot{a}(0) = -i\beta a - i\alpha a^+ \quad (47)$$

Thus, $a(t)$ is the solution of Eq. (46) with the initial conditions (47). Looking for a solution of the form

$$a(t) = A \cos \omega t + B \sin \omega t$$

we find

$$a = A; \quad -i\beta a - i\alpha a^+ = \omega B$$

so that

$$\begin{aligned} a(t) &= a \cos \omega t - \frac{i\beta}{\omega} a \sin \omega t - \frac{i\alpha}{\omega} a^+ \sin \omega t \\ &= \left(\cos \omega t - \frac{i\beta}{\omega} \sin \omega t \right) \cdot a - \frac{i\alpha}{\omega} \sin \omega t \cdot a^+ \end{aligned}$$

This proves (35). By taking the adjoint we get (36).

Finally it is clear that (37) and (38) are the limits of (35), (36) respectively as $\omega \rightarrow 0$ and that (38) satisfies equation (40) with initial condition $a^+(0) = a^+$.

Lemma 6 *Denote*

$$f(t) := \langle \psi_0, e^{itH} \psi_0 \rangle \quad (48)$$

If $\omega \neq 0$, then f satisfies the equation

$$\left(1 + \frac{\alpha^2}{\omega^2} \sin^2 \omega t \right) \frac{df}{dt} = -\frac{\alpha^2}{2\omega} \sin \omega t \left(\cos \omega t + \frac{i\beta}{\omega} \sin \omega t \right) f \quad (49)$$

with ω given by (34) and

$$f(0) = 1 \quad (50)$$

Proof. Let $f(t)$ be given by (48). We have

$$\frac{df}{dt} = \frac{i\alpha}{2} \langle \psi_0, e^{itH} a^{+2} \psi_0 \rangle = \frac{i\alpha}{2} \langle \psi_0, a^2 e^{itH} \psi_0 \rangle$$

Using the identity:

$$e^{itH} (a^+)^2 = e^{itH} a^+ e^{-itH} e^{itH} a^+ e^{-itH} e^{itH} = a^+(t) a^+(t) e^{itH} \quad (51)$$

and substituting $a^+(t)$ from Lemma (5), we get:

$$\begin{aligned}
\frac{df}{dt} &= \frac{i\alpha}{2} \langle \psi_0, a^+(t) a^+(t) e^{itH} \psi_0 \rangle = \\
&= \frac{i\alpha}{2} \langle \psi_0, \frac{i\alpha}{\omega} \sin \omega t \cdot a((\cos \omega t + \frac{i\beta}{\omega} \sin \omega t) a^+ \\
&\quad + \frac{i\alpha}{\omega} \sin \omega t \cdot a) e^{itH} \psi_0 \rangle \\
&= -\frac{\alpha^2}{2\omega} \sin \omega t \left(\cos \omega t + \frac{i\beta}{\omega} \sin \omega t \right) \langle \psi_0, a a^+ e^{itH} \psi_0 \rangle \\
&\quad - \frac{i\alpha^3}{2\omega^2} \sin^2 \omega t \langle \psi_0, a^2 e^{itH} \psi_0 \rangle
\end{aligned}$$

Since

$$\begin{aligned}
\langle \psi_0, a a^+ e^{itH} \psi_0 \rangle &= \langle \psi_0, (1 + a^+ a) e^{itH} \psi_0 \rangle \\
&= \langle \psi_0, e^{itH} \psi_0 \rangle = f(t)
\end{aligned}$$

and according to (51),

$$\langle \psi_0, a^2 e^{itH} \psi_0 \rangle = \frac{2}{i\alpha} \frac{df}{dt}$$

we get:

$$\begin{aligned}
\frac{df}{dt} &= -\frac{\alpha^2}{2\omega} \sin \omega t \left(\cos \omega t + \frac{i\beta}{\omega} \sin \omega t \right) f \\
&\quad - \frac{\alpha^2}{\omega^2} \sin^2 \omega t \frac{df}{dt}
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\left(1 + \frac{\alpha^2}{\omega^2} \sin^2 \omega t \right) \frac{df}{dt} \\
&= -\frac{\alpha^2}{2\omega} \sin \omega t \left(\cos \omega t + \frac{i\beta}{\omega} \sin \omega t \right) f
\end{aligned}$$

which is (49). The initial condition (50) is obvious.

Lemma 7 *If $\omega^2 \neq 0$, then the vacuum expectation of the evolution operator is*

$$\begin{aligned}
f(t) &= \langle \psi_0, e^{itH(\beta, \omega)} \psi_0 \rangle \\
&= \left(\frac{2e^{-i\beta t}}{(1 + \frac{\beta}{\omega})} e^{-it\omega} + (1 - \frac{\beta}{\omega}) e^{it\omega} \right)^{1/2} \\
&\Leftrightarrow f(t) = \langle \psi_0, e^{itH(\beta, \omega)} \psi_0 \rangle = \left(\frac{e^{-i\beta t}}{(\cos \omega t - i \frac{\beta}{\omega} \sin \omega t)} \right)^{1/2}
\end{aligned} \tag{52}$$

Proof. By direct calculations one can check that the function (52) is the solution of the differential equation (49) with the initial condition (50). In fact:

$$\begin{aligned}
f(t) &= \langle \psi_0, e^{itH(\beta, \omega)} \psi_0 \rangle \\
&= \left(\frac{2e^{-i\beta t}}{(1 + \frac{\beta}{\omega}) e^{-it\omega} + (1 - \frac{\beta}{\omega}) e^{it\omega}} \right)^{1/2} \\
f'(t) &= \frac{1}{2} \left(\frac{2e^{-i\beta t}}{(1 + \frac{\beta}{\omega}) e^{-it\omega} + (1 - \frac{\beta}{\omega}) e^{it\omega}} \right)^{-1/2} \\
&\quad \left(\frac{-i\beta 2e^{-i\beta t}}{(1 + \frac{\beta}{\omega}) e^{-it\omega} + (1 - \frac{\beta}{\omega}) e^{it\omega}} \right. \\
&\quad \left. - \frac{2e^{-i\beta t}(-(1 + \beta\omega)i\omega e^{-it\omega} + (1 - \beta\omega)i\omega e^{it\omega})}{(1 + \beta\omega)e^{-it\omega} + (1 - \beta\omega)e^{it\omega}} \right) \\
&= \frac{1}{2} \frac{1}{f(t)} \left(-i\beta f(t)^2 + i\omega f(t)^2 \frac{(1 + \frac{\beta}{\omega}) e^{-it\omega} - (1 - \frac{\beta}{\omega}) e^{it\omega}}{(1 + \frac{\beta}{\omega}) e^{-it\omega} + (1 - \frac{\beta}{\omega}) e^{it\omega}} \right) \\
&= \frac{i}{2} f(t) \left(-\beta + \omega \frac{(1 + \frac{\beta}{\omega}) e^{-it\omega} - (1 - \frac{\beta}{\omega}) e^{it\omega}}{(1 + \frac{\beta}{\omega}) e^{-it\omega} + (1 - \frac{\beta}{\omega}) e^{it\omega}} \right)
\end{aligned}$$

Using

$$e^{it\omega} = \cos \omega t + i \sin \omega t$$

one finds that

$$\left(1 + \frac{\beta}{\omega} \right) e^{-it\omega} + \left(1 - \frac{\beta}{\omega} \right) e^{it\omega} = e^{-it\omega} + \frac{\beta}{\omega} e^{-it\omega} + e^{it\omega} - \frac{\beta}{\omega} e^{it\omega}$$

$$\begin{aligned}
&= e^{it\omega} + e^{-it\omega} + \frac{\beta}{\omega} (e^{-it\omega} - e^{it\omega}) \\
&= 2 \cos \omega t + \frac{\beta}{\omega} (\cos \omega t - i \sin \omega t - \cos \omega t - i \sin \omega t) \\
&= 2 \left(\cos \omega t - i \frac{\beta}{\omega} \sin \omega t \right)
\end{aligned}$$

Similarly

$$\begin{aligned}
&\left(1 + \frac{\beta}{\omega}\right) e^{-it\omega} - \left(1 - \frac{\beta}{\omega}\right) e^{it\omega} = e^{-it\omega} + \frac{\beta}{\omega} e^{-it\omega} - e^{it\omega} + \frac{\beta}{\omega} e^{it\omega} \\
&= (e^{-it\omega} - e^{it\omega}) + \frac{\beta}{\omega} (e^{-it\omega} + e^{it\omega}) \\
&= -2i \sin \omega t + \frac{\beta}{\omega} 2 \cos \omega t = 2 \frac{\beta}{\omega} \left(\cos \omega t - i \frac{\omega}{\beta} \sin \omega t \right)
\end{aligned}$$

In conclusion

$$\begin{aligned}
f'(t) &= i/2 f(t) \left(-\beta + \omega \frac{-i \sin \omega t + \frac{\beta}{\omega} \cos \omega t}{\cos \omega t - i \frac{\beta}{\omega} \sin \omega t} \right) \Leftrightarrow \\
&\quad (\cos \omega t - i \frac{\beta}{\omega} \sin \omega t) f'(t) \\
&= \frac{i}{2} f(t) \left(-\beta \cos \omega t + i \frac{\beta^2}{\omega} \sin \omega t - i \omega \sin \omega t + \beta \cos \omega t \right) \Leftrightarrow \\
&\quad \Leftrightarrow = i/2 f(t) \left(i \frac{\beta^2}{\omega} \sin \omega t - i \omega \sin \omega t \right) \Leftrightarrow \\
&\quad \Leftrightarrow = -1/2 f(t) \left(\frac{\beta^2}{\omega^2} - 1 \right) \omega \sin \omega t
\end{aligned}$$

Therefore (52) is equivalent to

$$f(t) = \langle \psi_0, e^{itH(\beta, \omega)} \psi_0 \rangle = \left(\frac{e^{-i\beta t}}{\cos \omega t - i \frac{\beta}{\omega} \sin \omega t} \right)^{1/2}$$

Lemma 8 *If $\omega^2 = 0$, then the vacuum expectation of the evolution operator is*

$$f(t) = \langle \psi_0, e^{itH} \psi_0 \rangle = \frac{e^{-i\beta t/2}}{(1 - i\beta t)^{1/2}} \quad (53)$$

Proof. Repeating the proofs of Lemmata (5), (6) we find that

1. the Heisenberg evolutions of a , a^+ are given by:

$$a(t) = (1 - i\beta t)a - i\beta t a^+ \quad ; \quad a^+(t) = (1 + i\beta t)a^+ + i\beta t a$$

2. $f(t)$ satisfies the following differential equation:

$$(1 + \beta^2 t^2) \frac{df}{dt} = -\frac{\beta^2 t}{2} (i\beta t + 1) f$$

with the initial condition $f(0) = 1$.

One can check that (53) is the solution of this equation.

Remark. Notice, the the expression (53) is the limit of the expression (52) as $\omega \rightarrow 0$.

3 The Meixner distributions

In this section, following [12], [13] we recall some known facts about Meixner distributions and Meixner classes. For an historical discussion on their origins we refer to [14] or to the Appendix of [5].

3.1 The 5-th Meixner class

Definition 4 *A real valued random variable X is said to have Meixner distribution with parameters $a > 0$, $b \in (-\pi, \pi)$, $\mu \in \mathbb{R}$, $\delta > 0$ (or to belong to the 5-th Meixner class) if its density function is*

$$p_X(x; a, b, \delta, \mu) = \frac{(2 \cos(b/2))^{2\delta}}{2a\pi\Gamma(2\delta)} \exp\left(\frac{b(x - \mu)}{a}\right) \left| \Gamma\left(\delta + \frac{i(x - \mu)}{a}\right) \right|^2 \quad (54)$$

Lemma 9 *The characteristic function of the Meixner random variable X with parameters a, b, μ, δ is*

$$E(e^{itX}) = \left(\frac{\cos \frac{b}{2}}{\cosh \frac{at - ib}{2}} \right)^{2\delta} e^{i\mu t} \quad (55)$$

For the proof, see [12].

3.2 The 4–th Meixner class: Gamma distributions

Definition 5 *A real valued random variable X is said to have the Gamma distribution with parameters $a, \theta > 0, \mu \in \mathbb{R}$ (or to belong to the 3–d Meixner class) if its density function is*

$$p_X(x; a, \theta, \mu) = \frac{(x - \mu)^{a-1} e^{-(x-\mu)/\theta}}{\Gamma(a)\theta^a} \mathbf{1}_{[0, \infty)} \quad (56)$$

Lemma 10 *The characteristic function of the Gamma random variable X with parameters a, θ, μ is*

$$E(e^{itX}) = \frac{e^{-it\mu}}{(1 - i\theta t)^a} \quad (57)$$

Proof. It is known (for example, see [21]) that, denoting \mathcal{F} the Fourier transform,

$$\mathcal{F} \left[\frac{x^{a-1} e^{-x/\theta}}{\Gamma(a)\theta^a} \chi_{[0, \infty)}(x) \right] (t) = (1 - i\theta t)^{-a} \quad (58)$$

Moreover, for any ϕ , such that $\mathcal{F}[\phi]$ exists,

$$\mathcal{F}[\phi(x - \mu)](t) = e^{-it\mu} \mathcal{F}[\phi(x)](t) \quad (59)$$

Combining (58) and (59) we have (57).

3.3 The 3–d Meixner class: Negative Binomial (Pascal) distributions

Definition 6 *A real valued random variable X is said to have Negative Binomial (or Pascal) distribution with parameters $0 < p < 1, r \neq 0, \mu \in \mathbb{R}$,*

and $d \neq 0$ (or to belong to the 4-th Meixner class) if its probability density function is given by

$$P(x; r, p, \mu, d) = \sum_{n=0}^{\infty} \binom{r}{n} p^r (1-p)^n \delta(x - nd - \mu)$$

where, by definition, for any $r \in \mathbb{R}$, $k \in \mathbb{N} \cup \{0\}$:

$$\binom{r}{k} := \frac{r(r+1) \dots (r+k-1)}{k!} \quad ; \quad k \geq 1; \text{ and } \binom{r}{0} := 1$$

Remark. Recall, for completeness, that the 2-d Meixner class is the Poisson and the 1-st one is the Gaussian.

4 Meixner distributions and quadratic Bose Hamiltonians

Theorem 2 *Suppose that the condition*

$$\det h = -\omega^2 = \alpha^2 - \beta^2 > 0 \quad (60)$$

is satisfied. Then the vacuum expectation $f(t) = \langle \psi_0, e^{itH} \psi_0 \rangle$, with H given by (30), (32), is the characteristic function of the Meixner type 5 random variable with parameters

$$a = 2|\omega| \quad , \quad b = 2i \log \frac{\omega + \beta}{i\alpha} \quad , \quad \mu = -\frac{\beta}{2} \quad , \quad \delta = \frac{1}{4} \quad (61)$$

More explicitly:

$$\left(\frac{2e^{-i\beta t}}{\left(1 + \frac{\beta}{\omega}\right) e^{-it\omega} + \left(1 - \frac{\beta}{\omega}\right) e^{it\omega}} \right)^{1/2} = e^{it\beta/2} \left(\frac{\cos \frac{b}{2}}{\cosh \frac{at-ib}{2}} \right)^{2 \cdot \frac{1}{4}} \quad (62)$$

Proof. The idea of the proof is to fit the parameters a , b , μ and δ , given by (61) with those of the characteristic function given by Lemma 9. If $\omega^2 < 0$, then $\omega = -i|\omega|$. Hence,

$$e^{at/2} = e^{|\omega|t} = e^{-i\omega t} \quad (63)$$

Using the definition of ω , we have

$$(\omega + \beta)(\omega - \beta) = \omega^2 - \beta^2 = -\alpha^2 = (i\alpha)^2 \Leftrightarrow \frac{\omega + \beta}{i\alpha} = \frac{i\alpha}{\omega - \beta} \quad (64)$$

In particular from this and (61) it follows that:

$$e^{-ib/2} = e^{\log \frac{\omega + \beta}{i\alpha}} = \frac{\omega + \beta}{i\alpha} = \frac{i\alpha}{\omega - \beta} \Leftrightarrow e^{ib/2} = \frac{\omega - \beta}{i\alpha} \quad (65)$$

Using (65) and recalling that ω is purely imaginary and β, α are real, we find:

$$\cos \frac{b}{2} = \operatorname{Re} (e^{-ib/2}) = \frac{\omega}{i\alpha} \quad (66)$$

From (63,65) we have:

$$\begin{aligned} \cosh \frac{at - ib}{2} &= \frac{1}{2} (e^{at/2} e^{-ib/2} + e^{-at/2} e^{ib/2}) \\ &= \frac{1}{2} \left(e^{-i\omega t} \frac{\omega + \beta}{i\alpha} + e^{i\omega t} \frac{\omega - \beta}{i\alpha} \right) \\ &= \frac{1}{2} \frac{\omega}{i\alpha} \left(\left(1 + \frac{\beta}{\omega}\right) e^{-it\omega} + \left(1 - \frac{\beta}{\omega}\right) e^{it\omega} \right) \\ &= \frac{1}{2} \cos \frac{b}{2} \left(\left(1 + \frac{\beta}{\omega}\right) e^{-it\omega} + \left(1 - \frac{\beta}{\omega}\right) e^{it\omega} \right) \end{aligned} \quad (67)$$

Replacing (66) and (67) in the characteristic function (52), we obtain:

$$\begin{aligned} e^{i(\beta/2)t} \left(\cos \frac{b}{2} \right)^{2 \cdot \frac{1}{4}} \left(\cosh \frac{at - ib}{2} \right)^{-2 \cdot \frac{1}{4}} &= \\ = e^{-i\beta t/2} \left(\frac{\omega}{i\alpha} \right)^{-1/2} \left(\frac{1}{2} \right)^{-1/2} \left(\frac{\omega + \beta}{i\alpha} e^{-i\omega t} + \frac{\omega - \beta}{i\alpha} e^{i\omega t} \right)^{-1/2} \\ = \frac{\sqrt{2} e^{-i\beta t/2}}{\left(\left(1 + \frac{\beta}{\omega}\right) e^{-it\omega} + \left(1 - \frac{\beta}{\omega}\right) e^{it\omega} \right)^{1/2}} \end{aligned} \quad (68)$$

which is the vacuum expectation given by Lemma 7.

Definition 7 We say that two characteristic functions $\phi_1(t)$ and $\phi_2(t)$ are equal up to simple transformations if $\phi_1(t)$ can be transformed into $\phi_2(t)$ by applying a finite sequence of the following transformations:

"time shift"

$$\phi(t) \rightarrow \phi(t)e^{imt} \quad ; \quad m \in \mathbb{R},$$

"independent copying"

$$\phi(t) \rightarrow \phi(t)^q \quad ; \quad q > 0,$$

"time rescaling"

$$\phi(t) \rightarrow \phi(kt) \quad ; \quad k \neq 0.$$

Proposition 1 Up to simple transformations the characteristic function of any Meixner random variable of type V, can be reduced to a vacuum expectation of the form (52) for some homogeneous quadratic Bose Hamiltonian H satisfying (60).

Proof. Consider the Meixner type V random variable X with parameters (a, b, δ, μ) and put:

$$\alpha = \frac{a}{2}(\cos \frac{b}{2})^{-1} \tag{69}$$

$$\beta = \frac{a}{2} \tan \frac{b}{2} \tag{70}$$

Then the vacuum expectation of the quadratic Hamiltonian, specified by these parameters, is the characteristic function $\phi_{a', b', \delta', \mu'}(t)$ of Meixner distribution with parameters a' , b' , δ' , and μ' , where:

$$\begin{aligned} a' = 2|\omega| &= 2 \left| \sqrt{\left(\frac{a}{2} \tan \frac{b}{2}\right)^2 - \left(\frac{a}{2}(\cos \frac{b}{2})^{-1}\right)^2} \right| = \\ a \left| \sqrt{\tan^2 \frac{b}{2} - (\cos \frac{b}{2})^{-2}} \right| &= a \left| \sqrt{\frac{\sin^2(b/2) - 1}{\cos^2(b/2)}} \right| = a|i| = a \\ e^{\frac{-ib'}{2}} = \frac{\omega + \beta}{i\alpha} &= \frac{\frac{a}{2}\sqrt{\tan^2 \frac{b}{2} - (\cos \frac{b}{2})^{-2}} + \frac{a}{2} \tan \frac{b}{2}}{i\frac{a}{2}(\cos \frac{b}{2})^{-1}} = \end{aligned}$$

$$= \frac{i + \tan \frac{b}{2}}{i^{-1} \cos \frac{b}{2})^{-1}} = \cos \frac{b}{2} - i \sin \frac{b}{2} = e^{\frac{ib}{2}}$$

and, therefore, $b' = b$, $\mu' = \frac{1}{2}b$ and $\delta' = \frac{1}{4}$. For any δ, μ we have:

$$\phi_{a,b,\delta,\mu}(t) = e^{i\mu t} (e^{-i\mu' t} \phi_{a,b,1/4,\mu'}(t))^{4\delta}$$

This proves the statement.

Theorem 3 *Suppose that*

$$\det h = -\omega^2 = 0 \quad (71)$$

Then the vacuum expectation $f(t) = \langle \psi_0, e^{itH} \psi_0 \rangle$ is the characteristic function of the Gamma distribution with parameters

$$a = 1/2 \quad , \quad \theta = \beta \quad , \quad \mu = \beta/2$$

Proof. This follows from Lemmata (8) and (10).

Remark 2. It is easy to verify that the characteristic function of any Gamma distributed random variable, up to simple transformations, is the vacuum expectation of some Bose homogeneous quadratic Hamiltonian satisfying (71).

Theorem 4 *Suppose*

$$\det h = -\omega^2 < 0 \quad (72)$$

Then the vacuum expectation $f(t) = \langle \psi_0, e^{itH} \psi_0 \rangle$ is the characteristic function of the Negative Binomial distribution with parameters

$$r = 1/2 \quad , \quad p = \frac{2\omega}{\omega + \beta} \quad , \quad \mu = \frac{\beta - \omega}{2} \quad , \quad d = -2\omega \quad (73)$$

Proof. From Lemma 7 we have:

$$\begin{aligned} f(t) &= \left(\frac{2e^{-i\beta t}}{(1 + \frac{\beta}{\omega}) e^{-it\omega} + (1 - \frac{\beta}{\omega}) e^{it\omega}} \right)^{1/2} = \sqrt{2} e^{-i\beta t/2} \left(\frac{\omega + \beta}{\omega} e^{-i\omega t} - \frac{\beta - \omega}{\omega} e^{i\omega t} \right)^{-1/2} \\ &= \sqrt{2} e^{-it\frac{\beta - \omega}{2}} \left(\frac{\omega}{\omega + \beta} \right)^{1/2} \left(1 - \frac{\beta - \omega}{\beta + \omega} e^{2i\omega t} \right)^{-1/2} \end{aligned}$$

Note that

$$1 - \frac{2\omega}{\beta + \omega} = \frac{\beta - \omega}{\beta + \omega}$$

therefore, defining p , d , and μ through (73) we have:

$$f(t) = p^{1/2} e^{-i\mu t} \frac{1}{\sqrt{1 - (1-p)e^{-idt}}}$$

whose series expansion around $x = 0$ is:

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{\infty} c_n x^n \quad (74)$$

and the coefficients c_n are given by Newton's binomial:

$$c_n = \frac{1}{n!} \left. \frac{d^n}{dx^n} \frac{1}{\sqrt{1-x}} \right|_{x=0} = \frac{1}{n!} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \dots \cdot \frac{2n-1}{2} = \binom{1/2}{n} \quad ; n \geq 1$$

and $c_0 = 1$. Using this expansion, we get the series expansion:

$$f(t) = K e^{-i\mu t} \sum_{n=0}^{\infty} c_n (1-p)^n e^{-indt}$$

which, since $|e^{-indt}| = 1$, is uniformly convergent for any t and for any p in a bounded set. In our case since $\omega^2 > 0$, (73) implies that $-1 < 1-p = \frac{\beta-\omega}{\beta+\omega} < 1$, i.e. $|1-p| < 1$ and the series converges uniformly for all t . The inverse Fourier transform of f is

$$F^{-1}[f(t)](x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixt} f(t) dt = \frac{p^{1/2}}{2\pi} \int_{-\infty}^{\infty} e^{i(x-\mu)t} \sum_{n=0}^{\infty} c_n (1-p)^n e^{-indt} dt$$

Hence,

$$\begin{aligned} F^{-1}[f(t)](x) &= \frac{p^{1/2}}{2\pi} \sum_{n=0}^{\infty} c_n (1-p)^n \int_{-\infty}^{\infty} e^{i(x-\mu-nd)t} dt \\ &= \sum_{n=0}^{\infty} c_n p^{-1/2} (1-p)^n \delta(x-\mu-nd) = \sum_{n=0}^{\infty} \binom{1/2}{n} p^{-1/2} (1-p)^n \delta(x-\mu-nd) \end{aligned}$$

which is density function of the Pascal distribution with $r = 1/2$.

Proposition 2 *The characteristic function of any Negative Binomial distribution is, up to simple transformations, the vacuum expectation of some 1-dimensional homogeneous quadratic Bose Hamiltonian satisfying (71).*

Proof. Given $0 < p < 1$, $r \neq 0$, $\mu \in \mathbb{R}$, and $d \neq 0$, let the parameters α and β be:

$$\alpha = \sqrt{1 - \left(\frac{p}{2-p}\right)^2} \quad ; \quad \beta = 1 \quad (75)$$

Then the vacuum expectation of the quadratic Bose Hamiltonian, specified by these α and β is:

$$f(t) = p'^{1/2} e^{-i\mu't} \left(1 - (1-p')e^{-id't}\right)^{1/2}$$

where

$$\begin{aligned} \omega &= \sqrt{\beta^2 - \alpha^2} = \sqrt{1 - \left(1 - \left(\frac{p}{2-p}\right)^2\right)} = \frac{p}{2-p} \\ p' &= \frac{2\omega}{\omega + \beta} = \frac{p}{2-p} \left(\frac{p}{2-p} + 1\right)^{-1} = \frac{2p}{2-p} \left(\frac{2}{2-p}\right)^{-1} = p \\ \mu' &= \frac{1-\omega}{\omega} \quad ; \quad d' = -2\omega \end{aligned}$$

Let us apply the following (simple) transformation:

$$f(t) \rightarrow e^{i\mu t} (e^{-i\mu'\lambda t} f(\lambda t))^{2r}$$

where $\lambda = \frac{d}{d'}$. We have:

$$f(t) \rightarrow f_2(t) = p^r (1 - (1-p)e^{idt})^{-r} e^{i\mu t}$$

Applying the Newton binomial formula:

$$(1-x)^{-a} = \sum_{n=0}^{\infty} \binom{a}{n} x^n \quad (76)$$

and repeating computations of Theorem 4 one can find that $f_2(t)$ is the characteristic function of the distribution

$$\sum_{n=0}^{\infty} \binom{a}{n} p^r (1-p)^n \delta(x - dn - \mu)$$

Which is the negative Binomial distribution with parameters p , r , μ , and d .

We summarize Remarks 1-3 in the following theorem:

Theorem 5 *The characteristic function of any Meixner type III, IV, V distribution is, up to a simple transformation, the vacuum expectation of some 1-dimensional homogeneous quadratic Bose Hamiltonian.*

5 n -dimensional case

Let $a_i, a_i^+, i = 1, 2, \dots, n$ be Bose annihilation and creation operators, satisfying CCR:

$$[a_i, a_j] = [a_i^+, a_j^+] = 0 \quad ; \quad [a_i, a_j^+] = \delta_{i,j} \quad ; \quad i, j = 1, 2, \dots, n$$

Because of the commutativity of the creators (annihilators), the most general Hermitean quadratic expression in the a_i^\pm which is real homogeneous of degree 2 is:

$$H_{A,C} = \sum_{i,j=1}^n A_{ij} a_i^+ a_j^+ + \sum_{i,j=1}^n \bar{A}_{ij} a_i a_j + \sum_{i,j=1}^n C_{ij} a_i^+ a_j \quad (77)$$

where $A_{ij}, C_{ij} \in \mathbb{C}$. The Hermiteanity condition for $H_{A,C}$ and the mutual commutativity of creators (resp. annihilators) imply that

$$\begin{aligned} H_{A,C} = H_{A,C}^* &= \sum_{i,j=1}^n \bar{A}_{ij} a_j a_i + \sum_{i,j=1}^n A_{ij} a_j^+ a_i^+ + \sum_{i,j=1}^n \bar{C}_{ij} a_j^+ a_i \\ &= \sum_{i,j=1}^n A_{ji} a_i^+ a_j^+ + \sum_{i,j=1}^n \bar{A}_{ji} a_i a_j + \sum_{i,j=1}^n \bar{C}_{ji} a_i^+ a_j \end{aligned} \quad (78)$$

Therefore

$$\begin{aligned} H_{A,C} &= \frac{1}{2}(H_{A,C} + H_{A,C}^*) = \\ &= \sum_{i,j=1}^n \frac{1}{2}(A_{ij} + A_{ji}) a_i^+ a_j^+ + \sum_{i,j=1}^n \frac{1}{2}(\bar{A}_{ij} + \bar{A}_{ji}) a_i a_j + \sum_{i,j=1}^n \frac{1}{2}(C_{ij} \\ &\quad + \bar{C}_{ji}) a_i^+ a_j \end{aligned} \quad (79)$$

Therefore one can suppose that

$$A_{ij} = A_{ji} \quad ; \quad C_{ij} = \bar{C}_{ji}$$

i.e. that the $n \times n$ matrices $A := (A_{ij})$ and $C := (C_{ij})$ are respectively symmetric and Hermitean:

$$A = A^T \quad ; \quad C = C^*$$

Denote the n -component vectors (a_i) and (a_i^+) by a and a^* . Then, up to an additive constant, one can rewrite the Hamiltonian (77) in matrix form:

$$H_{A,C} = \frac{1}{2} (a^* A a^* + a A a) + a^* C a = \frac{1}{2} (a^+, a) \begin{pmatrix} C & A \\ \bar{A} & \bar{C} \end{pmatrix} \begin{pmatrix} a \\ a^+ \end{pmatrix} = H \quad (80)$$

Theorem 6 *There exist $n \times n$ matrices $\Phi(t)$ and $\Psi(t)$ such that:*

$$\begin{pmatrix} \Phi(t) & \Psi(t) \\ \bar{\Psi}(t) & \bar{\Phi}(t) \end{pmatrix} = \exp \left(it \frac{1}{2} \begin{pmatrix} C & A \\ \bar{A} & \bar{C} \end{pmatrix} \right)$$

Proof. This result was proved by Friedrichs and extended by Berezin ([9] pg. 122).

We denote ψ_0 the vacuum vector, characterized by $a_j \psi_0 = 0$ for all $j = 1, 2, \dots, n$.

Theorem 7

$$\langle \psi_0, e^{itH_{A,C}} \psi_0 \rangle = \frac{1}{\sqrt{\det \Phi(t) e^{iCt}}} = \det(\Phi(t))^{-1/2} \det(e^{iCt})^{-1/2}$$

Proof. This result was proved by Friedrichs and extended by Berezin ([9] pg. 122).

Lemma 11 *$\Phi(t)$ satisfies the following equation:*

$$A^{-1} \ddot{\Phi} - i[C, A^{-1}] \dot{\Phi} + (CA^{-1}C - A)\Phi = 0 \quad (81)$$

with the initial conditions $\Phi(0) = 1$, $\dot{\Phi}(0) = -iC$.

Proof. From the definition it follows that

$$\frac{d}{dt} \begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix} = i \begin{pmatrix} -C & -A \\ A & C \end{pmatrix} \begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix} \quad (82)$$

In particular,

$$\dot{\Psi} = -iC\Psi - iA\bar{\Phi} \quad (83)$$

$$\dot{\bar{\Phi}} = iA\Psi + iC\bar{\Phi} \quad (84)$$

Let us express Ψ using (84):

$$\Psi = \frac{1}{i}A^{-1} \left(\dot{\bar{\Phi}} - iC\bar{\Phi} \right) \quad (85)$$

Substituting this for Ψ into (83) we obtain:

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{i}A^{-1} \left(\dot{\bar{\Phi}} - iC\bar{\Phi} \right) \right) &= -iC \left(\frac{1}{i}A^{-1} \left(\dot{\bar{\Phi}} - iC\bar{\Phi} \right) \right) - iA\bar{\Phi} \\ -iA^{-1}\ddot{\bar{\Phi}} + (CA^{-1} - A^{-1}C)\dot{\bar{\Phi}} + i(A - CA^{-1}C)\bar{\Phi} &= 0 \end{aligned}$$

Multiplying by i and taking the adjoint, we have:

$$A^{-1}\ddot{\Phi} + i(CA^{-1} - A^{-1}C)\dot{\Phi} + (CA^{-1}C - A)\Phi = 0$$

which is (81). The initial conditions are clear.

5.1 Vector valued Meixner random variables

Recall that a random variable X , with values in a finite dimensional real vector space V (identified to its dual space) is defined by a linear map

$$X : V \rightarrow \text{real valued random variables}$$

An homogeneous quadratic Bose Hamiltonian H of the form (77), is uniquely determined by a pair (A, C) where A is a symmetric complex $d \times d$ matrix and C a complex self-adjoint matrix. The set of such pairs has a natural structure of finite dimensional real vector space, that we denote V_d , and the map

$$(A, C) \mapsto H_{A,C} \in \{\text{self-adjoint operators}\}$$

is clearly real linear. It is known that each $H_{A,C}$ can be identified to a real valued classical random variable with respect to the vacuum vector.

Definition 8 *In the above notations, a V_d -valued random variable X is called of Meixner type if for each $(A, C) \in V_d$ the characteristic function of $X_{(A,C)}$ coincides with the vacuum characteristic function of $H_{A,C}$:*

$$E(e^{itX_{(A,C)}}) = \langle \psi_0, e^{itH_{A,C}} \psi_0 \rangle$$

where $H_{A,C}$ denotes the homogeneous quadratic Fock Bose Hamiltonian with d -degrees of freedom, ψ_0 the corresponding vacuum vector and E the expectation with respect to the random variable $X_{(A,C)}$.

Remark. From Theorem (6) we then know that Definition (8) is equivalent to require that

$$E(e^{itX_{(A,C)}}) = \det(\Phi(t) \cdot e^{iCt})^{-1/2} \quad (86)$$

where $\Phi(t)$ is the solution of equation (81).

5.2 The case of commuting A and C

Theorem 8 Let the $d \times d$ Hermitean complex matrices A , C , $\Phi(t)$ be as in Theorem (6) and suppose that

$$[A, C] = 0 \quad (87)$$

or, equivalently, that A and C can be simultaneously diagonalized by an orthogonal transformation Q :

$$Q^T A Q = \Lambda_A = \text{diag}[\alpha_1, \alpha_2, \dots, \alpha_n] \quad (88)$$

$$Q^T C Q = \Lambda_C = \text{diag}[\beta_1, \beta_2, \dots, \beta_n] \quad (89)$$

Then

$$\begin{aligned} (\det \Phi e^{iCt})^{-1/2} &= \left(\frac{e^{-it(\beta_1 + \beta_2 + \dots + \beta_n)}}{\prod_{j=1}^n \left(\cos \omega_j t - \frac{i\beta_j}{\omega_j} \sin \omega_j t \right)} \right)^{1/2} = \\ &= \prod_{j=1}^n \left(\frac{e^{-it\beta_j}}{\cos \omega_j t - \frac{i\beta_j}{\omega_j} \sin \omega_j t} \right)^{1/2} \end{aligned} \quad (90)$$

where

$$\omega_i^2 = \beta_i^2 - \alpha_i^2 \quad (91)$$

In particular the vacuum distribution of the symmetric operator $H_{A,C}$, given by (80) is a product of Meixner distributions.

Proof. Since A and C are commuting symmetric real matrices with eigenvalues $\alpha_i, \beta_i \in \mathbb{R}$ respectively, then

$$(\det \Phi e^{iCt})^{-1/2} = (\det \Phi)^{-1/2} (\det e^{iCt})^{-1/2}$$

Notice that $\det Q = 1$. Hence,

$$\det e^{iCt} = \det e^{itQ^T \Lambda_C Q} = \det (Q^T e^{it\Lambda_C} Q) = \det e^{it\Lambda_C} = \prod_{j=1}^n e^{it\beta_j} = e^{it(\beta_1 + \beta_2 + \dots + \beta_n)} \quad (92)$$

Secondly, denote $\Phi' = Q^T \Phi Q$. Then Eq. (81) in terms of Φ' becomes

$$\ddot{\Phi}' + (\Lambda_C^2 - \Lambda_A^2)\Phi' = 0$$

Denote the components of the Φ' matrix by ϕ_{ij} , $i, j = 1, 2, \dots, n$. The equation for each component is independent. For the off-diagonal components we have:

$$\frac{d^2}{dt^2}\phi_{ij} = 0 \quad ; \quad \phi_{ij}(0) = 0 \quad ; \quad \frac{d}{dt}\phi_{ij}(0) = 0 \quad ; \quad i \neq j$$

With the obvious solution $\phi_{ij}(t) = 0$. For the diagonal components we have:

$$\frac{d^2}{dt^2}\phi_{ii} + \omega_i^2 \phi_{ii} = 0 \quad ; \quad \phi_{ii}(0) = 1 \quad ; \quad \dot{\phi}_{ii}(0) = -i\beta_i$$

where ω_i is given by (91). Solving this equation, we have:

$$\phi_{ii}(t) = \cos \omega_i t - \frac{i\beta_i}{\omega_i} \sin \omega_i t$$

Thus, we see that Φ' is a diagonal matrix and

$$\det \Phi' = \prod_{j=1}^n \phi_{jj}(t) = \prod_{j=1}^n \left(\cos \omega_j t - \frac{i\beta_j}{\omega_j} \sin \omega_j t \right) \quad (93)$$

Finally, note that

$$\det \Phi' = \det Q^T \Phi Q = \det \Phi \quad (94)$$

Combining (92), (93), and (94) we obtain (90).

5.3 The 1-dimensional case

Consider the simplest quantum model with the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ and the pair of creation and annihilation operators a^+ and a satisfying

$$[a, a^+] = 1$$

Denote $\psi_0 \in \mathcal{H}$ the vector (vacuum) such that

$$a\psi_0 = 0$$

We will determine the explicit form of the right hand side of (7) when $n = 1$. In this case A and C are real numbers and $\Phi(t)$ is a complex valued function.

Theorem 9 *If $n = 1$ then the following identity holds*

$$(\Phi(t)e^{iCt})^{-1/2} = \sqrt{2} \frac{e^{-iCt/2}}{\left[\left(1 + \frac{C}{\omega}\right) e^{-i\omega t} + \left(1 - \frac{C}{\omega}\right) e^{i\omega t}\right]^{1/2}} \quad (95)$$

where

$$\omega = \sqrt{C - |A|^2} \quad (96)$$

Proof. Let us verify the equality (95). From Theorem (6) one gets

$$\begin{pmatrix} \dot{\Phi} & \dot{\Psi} \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix} = ih \begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix} \quad (97)$$

or

$$\begin{aligned} \dot{\Phi} &= -iC\Phi - iA\bar{\Psi} \\ \dot{\Psi} &= -iC\Psi - iA\bar{\Phi} \end{aligned} \quad (98)$$

The initial boundary conditions are

$$\Phi(0) = 1, \quad \dot{\Phi}(0) = -iC; \quad \Psi(0) = 0, \quad \dot{\Psi}(0) = -iA \quad (99)$$

From this one can show that $\Phi(t)$ has the form

$$\Phi(t) = \frac{1}{2} \left[\left(1 + \frac{C}{\omega}\right) e^{-i\omega t} + \left(1 - \frac{C}{\omega}\right) e^{i\omega t} \right]$$

and this proves the relation (95).

Acknowledgments This work was supported by the grant of the Russian Science Foundation RSF 14-11-00687.

References

- [1] L. Accardi (1), I.Ya. Aref'eva, I.V.Volovich,
Non isomorphism of the Bose and Fermi realization of $sl(2, \mathbb{R})$. in preparation (2014).
- [2] L. Accardi, H.-H. Kuo, and A. Stan:
Moments and commutators of probability measures,
Infinite Dimensional Analysis, Quantum Probability and Related Topics
Quantum Probability Communications, 10 (2007) 591-612
- [3] Accardi L., Boukas A., Franz U.: Renormalized powers of quantum white noise, IDA-QP (Infinite Dimensional Anal. Quantum Probab. Related Topics) 9 (1) (2006) 129–147 MR2214505, DOI: 10.1142/S0219025706002263 Preprint Volterra n. 597 (2006)
- [4] L. Accardi, Andreas Boukas:
Higher Powers of q -Deformed White Noise,
Methods of Functional Analysis and Topology 12 (3) (2006) 205–219
- [5] Luigi Accardi, Andreas Boukas:
White noise calculus and stochastic calculus,
in: Stochastic Analysis: Classical and Quantum, T. Hida, K. Saito (eds.)
World Scientific (2005) 260–300
Proceedings International Conference on "Stochastic analysis: classical and quantum, Perspectives of white noise theory",
Meijo University, Nagoya, 1-5 November 2004,
Preprint Volterra n. 579 (2005)
- [6] Accardi L., Franz U., Skeide M.:
Renormalized squares of white noise and other non- Gaussian noises as Levy processes on real Lie algebras,
Comm. Math. Phys. 228 (2002) 123–150
Preprint Volterra, N. 423 (2000)
- [7] Accardi L., Lu Y.G., Volovich I.V.: White noise approach to classical and quantum stochastic calculi, Lecture Notes of the Volterra-CIRM

International School with the same title, Trento, Italy, 1999,
Volterra Preprint N. 375 July (1999)

- [8] Accardi L., Schurmann M., von Waldenfels W.:
Quantum independent increment processes on superalgebras,
Math. ZeitSchr. 198 (4) (1988) 451-477
Preprint, Heidelberg, Stochastische Mathematische Modelle, January
(1987)
- [9] F. A. Berezin,
The Method of Second Quantization, Pure Appl. Phys. 24, Academic
Press, New York, 1966.
- [10] Friedrichs, K.O (1951): Mathematical aspects of the quantum theory
of fields, Commu Pure Appl. Math. 4, 161-224 (1951), 5 1-56, 349-411
(1952), 6, 1-72 (1953) (Collectively reissued: Interscience, New York,
1953)
- [11] I.S. Gradshteyn and I.M. Ryzhik: Table of Integrals, Series, and Prod-
ucts, Seventh Edition Hardcover by Alan Jeffrey (Editor), Daniel Zwill-
inger (Editor) Academic Press (2007)
- [12] B. Grigelionis Generalized z -distributions and related stochastic pro-
cesses Lithuanian Math. J. 41 (2001) 303–319
- [13] B. Grigelionis Processes of Meixner type Lithuanian Math. J. 39 (1999)
33–41
- [14] J. Meixner, Orthogonal Polynomsysteme mit einer besonderen gestalt
der erzeugenden funktion. (Orthogonal polynomial systems with a gen-
erating function of a special form.) J. London Math. Soc. 9 (1934) 6–13
(Zbl. 7, 307).
- [15] Parthasarathy K.R., Schmidt K.: Positive definite kernels continuous
tensor products and central limit theorems of probability theory, Springer
Lecture Notes in Mathematics no. 272 (1972)
- [16] Schürmann M.: White noise on bialgebras, Springer LNM 1544 (1993)
- [17] P. Śniady:
Quadratic bosonic and free white noises,
Commun. Math. Phys. 211 (3) (2000) 615–628 Preprint (1999)

- [18] Gabriela Popa, Aurel I. Stan: Two-dimensional Meixner random vectors and their semi-quantum operators. preprint November 2014
- [19] J. Schwinger, U.S Atomic Energy Commission Report NYO-3071, 1952 or D. Mattis, The Theory of Magnetism, Harper and Row, (1982)
- [20] R. Anishetty, M. Mathur and I. Raychowdhury: Irreducible $SU(3)$ Schwinger Bosons, arXiv:0901.0644.
- [21] Eric W. Weisstein: "Gamma Distribution." From MathWorld – A Wolfram Web Resource. <http://mathworld.wolfram.com/GammaDistribution.html>
- [22] <http://mathworld.wolfram.com/NegativeBinomialDistribution.html>